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EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE nD NAVIER-STOKES EQUATION

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Abstract

A fundamental problem in mathematics is to decide whether such smooth, physically reasonable solutions exist for the Navier-Stokes equations. We restrict attention here to incompressible fluids filling all of R^n. The Cauchy problem for the nD Navier-Stokes equations is reduced to the integral equations of Volterra and Volterra-Abel, investigation of which, allows us to solve positively question on uniqueness and smoothness of the solution.

1. Introduction

It is known that the Navier-Stokes equations are important for investigation of properties of fluid motion and difficult for qualitative analysis (see for instance, Caffarelli et al. [3] or Schlichting [9] and others). These equations are to be solved for an unknown velocity vector $v(x, t) = [v_1(x, t), ..., v_n(x, t)]$ and pressure $P(x, t)$, $(x, t) \in T_* = R^n \times$

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$(R_+ = [0, \infty))$. The Navier-Stokes equations are then given by [12]:

$$v_{it} + \sum_{j=1}^{n} v_j v_{ix_j} = f_i - \frac{1}{\rho} P_{x_i} + \mu \Delta v_i, \quad (i = \overline{1, n}), \tag{1.1}$$

$$div\, v = 0, \quad \forall (x, t) \in T_*, \quad (T = R^n \times [0, T_0]), \tag{1.2}$$

$$v_i|_{t=0} = v_{0i}(x_1, ..., x_n), \quad (x_1, ..., x_n) \in R^n, \tag{1.3}$$

where $f_i(x, t)$ are the components of a given, externally applied force (e.g., gravity), μ is a positive coefficient (the viscosity $\mu > 0$), and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables. The Euler equations are equations (1.1), (1.2) with *m* set equal to zero. There are many fascinating problems and conjectures about the behavior of solutions of the Euler equations (see Beale et al. [1], Constantin [2], Schlichting [9] or Scheffer [10]).

Starting with Leray [5], an important progress has been made in understanding weak solutions of the Navier-Stokes equations. To arrive at the idea of a weak solution of a PDE, one integrates the equation against a test function and then integrates by parts (formally) to make the derivatives fall on the test function. The partial regularity theorem of [3, 6] concerns a parabolic analogue of the Hausdorff dimension of the singular set of a suitable weak solution of Navier-Stokes problems.

Standard methods from PDE appear inadequate to settle the problem (see Fefferman [12]). Therefore, starting with paper [7], an important progress has been made in understanding smoothness of a solution of the 3D Navier-Stokes equations in $G^2_{n=3;\lambda}(D_0)$. Note, by theorem of Sobolev [11], the suitable solutions of Navier-Stokes problem in space $W^2_{n=3;\lambda}(D_0)$ were constructed in [7, 8], that is, the 3D Navier-Stokes system admits a unique global-in-time weak solution.

This is the first of a series of papers devoted to the initial value problem for the nD Navier-Stokes system of incompressible fluids. In the present paper, we establish the existence and uniqueness of a solution of the nD Navier-Stokes system in $G_n^1(D_0 = R^n \times (0, T_0))$, where the norm of this space is defined by the following:

$$\| \nu \|_{G_n^1(D_0)} = \sum_{i=1}^{n} \| v_i \|_{\tilde{G}^1(D_0)}$$

$$= \sum_{i=1}^{n} \left\{ \sum_{0 \le |k| \le 2} \| D^k v_i \|_{C(T)} + \sup_{R^n} \int_0^{T_0} | v_{it}(x_1, ..., x_n, t) | dt \right\}.$$

It is known that the turbulence solutions are conditional smooth in analytical sense [4, 9] for $0 < \mu < 1$. Therefore, we consider a class of suitable solutions of the Navier-Stokes problem in weight space of Sobolev's type $W_{n;\lambda}^2(D_* = R^n \times (0, \infty))$ with the norm

$$\begin{cases} \| \nu \|_{W_{n,\lambda}^2(D_*)} = \sum_{i=1}^{n} \| v_i \|_{\tilde{W}_\lambda^2(D_*)} \\ \qquad = \sum_{i=1}^{n} \left\{ \sum_{0 \le |k| \le 2} \sup_{R^n} \int_0^{\infty} \lambda(t) [D^k v_i(x_1, ..., x_n, t)]^2 dt \right. \\ \qquad\qquad \left. + \sup_{R^n} \int_0^{\infty} \lambda(t) [v_{it}(x_1, ..., x_n, t)]^2 dt \right\}^{\frac{1}{2}}; \\ 0 \le \lambda(t) : \int_0^{\infty} \lambda(t) t^{j-1} dt = q_j, \quad (j = \overline{0, 2}). \end{cases}$$

Our result about the nD Navier-Stokes system concerns the Cauchy problem with certain initial data. In the course of writing this paper, we discovered a natural restriction on the initial data of the Navier-Stokes system, which we refer to as the necessary conditions of resolvability in $G_n^1(D_0)$, $W_{n,\lambda}^2(D_*)$. It turns out to play a fundamental role in the mathematical theory of the Navier-Stokes systems. We do not attempt here to review the vast literature on the Navier-Stokes system, since there are fundamental works in this area (see for instance, Caffarelli et al. [3], Schlichting [9] or Scheffer [10] and others).

2. The Strict Solution of the Navier-Stokes Equation with Viscosity

There are various mathematical transformations in the theory of the differential equations in partial derivatives which simplify investigated problems and allow us to find the solutions in certain spaces [3, 5, 6, 9]. For $\mu > 0$, we show that the Navier-Stokes problem can be transformed to inhomogeneous linear equations of heat conductivity type under Cauchy condition that has the strict solution, consequently, the nD Navier-Stokes problem has the strict solution in space $W^2_{n;\lambda}(D_*)$. At least, this solution answers to mathematical question, and allows to construct the solution of the Navier-Stokes problem (1.1)-(1.3) for an incompressible fluid with viscosity.

Let us restrict attention to forces f and initial conditions ν_0 that satisfy

$$\begin{cases} v_i|_{t=0} = v_{0i}(x_1, \ldots, x_n) \equiv \lambda_i \vartheta_0(x_1, \ldots, x_n), \quad (i = \overline{1, n}), \\ div f \neq 0; \quad \nu_0 = (v_{01}, \ldots, v_{0n}), \end{cases} \tag{2.1}$$

where $0 < \lambda_i$, $(i = 1, \ldots, n)$ are fixed constants. Then speed components ν are defined by the rules

$$\begin{cases} v_i = \lambda_i \vartheta(x_1, \ldots, x_n, t), \quad (x_1, \ldots, x_n, t) \in T_*, \quad (i = \overline{1, n}), \\ \vartheta|_{t=0} = \vartheta_0(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in R^n, \\ div\, \nu = 0: \sum_{j=1}^{n} \lambda_j \vartheta_{x_j} = 0; \quad \sum_{j=1}^{n} v_j v_{ix_j} = \lambda_i \vartheta \sum_{j=1}^{n} \lambda_j \vartheta_{x_j} = 0. \end{cases} \tag{2.2}$$

Hence, the system (1.1) is transformed to the system of inhomogeneous linear equations

$$\lambda_i \vartheta_t = f_i - \frac{1}{\rho} P_{x_i} + \mu \lambda_i \Delta \vartheta, \quad (i = \overline{1, n}). \tag{2.3}$$

Here ϑ is a new unknown function which, by (2.2), defines the solution of the Navier-Stokes problem. To solve the system (2.3) at first, we define a pressure P. From system (2.3), by conditions (2.1), (2.2) and "*Algorithm Poissonization System* - *APS*" ([7]: for this, we take the partial derivations of the system (2.3) at x_i, and summarizing the obtained equations by the formula (1.2)), we obtain

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(2.3): \quad \frac{1}{\rho}\Delta P = -\theta_n F_0, \\ \theta_n = 2(n-2)\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1}\sqrt{\pi^n}; \quad F_0 \equiv -(\theta_n)^{-1}\sum_{i=1}^{n} f_{ix_i}(x_1, ..., x_n, t), \quad (n \geq 3), \\ \frac{1}{\rho} P = \int_{R^n} F_0(s_1, ..., s_n, t)\frac{ds_1 \cdots ds_n}{r^{n-2}}, \\ \qquad \left(r = \sqrt{\sum_{i=1}^{n}(x_i - s_i)^2};\ s_i - x_i = \tau_i;\ i = \overline{1, n}\right), \\ \frac{1}{\rho} P_{x_i} = \int_{R^n} \frac{\tau_i (n-2) F_0(x_1 + \tau_1, ..., x_n + \tau_n; t)\, d\tau_1 \cdots d\tau_n}{\sqrt{(\tau_1^2 + \cdots + \tau_n^2)^n}} \equiv \phi_i(x_1, ..., x_n, t). \end{cases} \tag{2.4}$$

Thus, by (2.4), the system (2.3) is equivalent to

$$\begin{cases} \vartheta_t = \Phi_0(x_1, ..., x_n, t) + \mu \Delta \vartheta, \\ \vartheta|_{t=0} = \vartheta_0(x_1, ..., x_n), \\ (\lambda_1)^{-1}(f_1 - \phi_1) = (\lambda_2)^{-1}(f_2 - \phi_2) = \cdots = (\lambda_n)^{-1}(f_n - \phi_n) \equiv \Phi_0. \end{cases} \tag{2.5}$$

It means that system (2.3) is transformed to the inhomogeneous linear equations of heat conductivity with a condition of Cauchy (2.5). It is well-known that the problem (2.5) with enough smooth initial data is decidable [11]. Namely, the strict solution is

$$\vartheta = \frac{1}{\sqrt{\pi^n}} \int_{R^n} \exp\left(-\left(\sum_{i=1}^{n} \tau_i^2\right)\right) \vartheta_0(x_1 + 2\tau_1\sqrt{\mu t}, ..., x_n + 2\tau_n\sqrt{\mu t})\, d\tau_1 \cdots d\tau_n$$

$$+\frac{1}{\sqrt{\pi^n}}\int_0^t\int_{R^n}\exp\left(-\left(\sum_{i=1}^n\tau_i^2\right)\right)\Phi_0(x_1+2\tau_1\sqrt{\mu(t-s)},\ ...,\ x_n$$

$$+2\tau_n\sqrt{\mu(t-s)};\ s)\times d\tau_1\cdots d\tau_n ds$$

$$\equiv\Psi(x_1,\ ...,\ x_n,\ t),\quad (s_i-x_i=2\tau_i\sqrt{\mu t};\ s_i-x_i=2\tau_i\sqrt{\mu(t-s)}),\qquad (2.6)$$

Ψ is a known function. The solution (2.6) has the same properties of the usual solution of heat conductivity equation. For example, throughout and for definiteness, we always assume that the following assumptions are satisfied:

$$\begin{cases}\sup\limits_{R^n}|\ D^k\vartheta_0\ |\le\beta_1;\quad \sup\limits_{T_*}|\ D^k\Phi_0(x_1,\ ...,\ x_n,\ t)\ |\le K_0Q(t)\le\beta_2,\quad (k=\overline{0,\ 2}),\\ [12],\ Q(t):K_0\int_0^\infty Q(s)ds\le\beta_3;\quad \int_0^\infty\lambda(t)t^{j-1}dt=q_j,\\ \qquad (j=\overline{1,\ 2};\ 0<\mu<\eta_0<\infty),\\ \sup\limits_{T_*}\frac{1}{\sqrt{\pi^n}}\int_{R^n}\exp(-(\tau_1^2+\cdots+\tau_n^2))|D^k\vartheta_0(\bar{l}_1,\ ...,\ \bar{l}_n)|d\tau_1\cdots d\tau_n\le\beta_1,\\ \sup\limits_{T_*}\frac{1}{\sqrt{\pi^n}}\int_0^t\int_{R^n}\exp(-(\tau_1^2+\cdots+\tau_n^2))|D^k\Phi_0(l_1,\ ...,\ l_n;\ s)|d\tau_1\cdots d\tau_n ds\le\beta_3,\\ \sup\limits_{R^n}\frac{1}{\sqrt{\pi^n}}\int_{R^n}\exp\left(-\left(\sum_{i=1}^n\tau_i^2\right)\right)\left\{\sum_{i=1}^n|\ \tau_j\ |\times|\vartheta_{0\bar{l}_j}(\bar{l}_1,\ ...,\ \bar{l}_n)|\right\}d\tau_1\cdots d\tau_n\le n\sqrt{2^{-1}n}\beta_1,\\ \frac{1}{\sqrt{\pi^n}}\int_0^t\int_{R^n}\exp\left(-\left(\sum_{i=1}^n\tau_i^2\right)\right)\frac{1}{\sqrt{t-s}}\sum_{j=1}^n|\ \tau_j\ |\\ \times|\Phi_{0l_j}(l_1,\ ...,\ l_n;\ s)|d\tau_1\cdots d\tau_n ds\le n\sqrt{2n}\beta_2\sqrt{t}=\beta_5\sqrt{t};\\ \left(\sup\limits_{R^n}\int_0^\infty\lambda(s)|\ \Phi_0(x_1,\ ...,\ x_n,\ s)\ |^2ds\right)^{\frac{1}{2}}\le\beta_2\sqrt{q_1},\\ \beta_4=n\sqrt{2^{-1}n}\beta_1;\quad \beta=\max\limits_{1\le i\le 5}\beta_i;\quad (\sqrt{\mu q_0}+\sqrt{q_1}+\sqrt{\mu q_2})\beta\le\overline{\beta}_0.\end{cases}$$

(2.7)

Then the solution (2.6) of problem (2.5) is bounded by norm of the space $\tilde{W}_\lambda^2(D_*)$ and we get estimations

$$\|\vartheta\|_{\tilde{W}_\lambda^2(D_*)}\le m_k(\beta_1+\beta_3)\sqrt{q_1}+\overline{\beta}_0\le 2m_k\beta\sqrt{q_1}+\overline{\beta}_0=K_*,\quad (0<m_k=\text{const}).$$

(2.8)

Hence, by transformation (2.2), we have the strict solution of system (1.1) which satisfies the condition (1.2). According to (2.1) and (2.7), there is a conditional smooth and unique solution of the Navier-Stokes problem in $W^2_{n;\lambda}(D_*)$ that is defined by (2.2), at that

$$\| \nu \|_{W^2_{n,\lambda}(D_*)} = \sum_{i=1}^{n} \| v_i \|_{\tilde{W}^2_\lambda(D_*)} = \sum_{i=1}^{n} \lambda_i \| \vartheta \|_{\tilde{W}^2_\lambda(D_*)} \leq d_0 K_*, \quad \left(d_0 = \sum_{i=1}^{n} \lambda_i \right). \tag{2.9}$$

Theorem 1 (Weak solutions to the Navier-Stokes system)**.** *Consider the Cauchy problem* (2.5) *for linear equations of heat conductivity type associated with the Navier-Stokes system* (1.1) *by* (2.2), *when the spatial domain* $T_* = R^n \times R_+$. *Assume that the initial data satisfy conditions* (2.1), (2.7) *and finally, take place* (2.8), (2.10). *Then the Cauchy problem for the Navier-Stokes equation* (1.1) *admits a global-in-time weak solution in* $\tilde{W}^2_\lambda(D_*)$. *Therefore, the Navier-Stokes system has the unique solution in* $W^2_{n;\lambda}(D_*)$.

Remark. Results of the specified point are connected with condition (2.1). It is known that not always the initial data is satisfied with the given condition. Therefore, in Section 3, we consider Navier-Stokes system (1.1) with the general condition (1.3).

3. Fluid with Small Viscosity

In Section 3, we consider the Navier-Stokes system (1.1) and we establish an existence theory and uniqueness of a solution for the Navier-Stokes system, when the initial data have certain restrictions. With that end in view, let us consider updating of the basic method (2.2), when

$$\begin{cases} f_i \equiv \lambda_i f_0(x_1, \ldots, x_n, t) + \varphi_{i\delta_0}(x_1, \ldots, x_n, t), \\ (x_1, \ldots, x_n, t) \in T = R^n[0, T_0], \\ |\varphi_{i\delta_0}| \le C_0\sqrt{\delta_0}, \\ (i = 1, \ldots, n;\ 0 < C_0 = \text{const};\ 0 < \delta_0 = \text{const} << 1), \end{cases} \tag{3.1}$$

here δ_0, $0 < \lambda_i$, $(i = 1, \ldots, n)$ are fixed constants. We see that the components of speed grow faster than the components of speed in (2.2) by the regulatory functions Ω_i, $(i = 1, \ldots, n)$, where $0 < \mu < 1$.

There are the various partial experimental methods connecting speed and pressure. Here we offer the method using regulatory functions for getting connections between pressure and speed, and this distribution law allows us to express speed in the integral form, when $\nu \in R^n$.

Offered method of integral transformations reduces the nonlinear Navier-Stokes problem to inhomogeneous linear problem of heat conductivity. In this case, problem of heat conductivity is reduced to the system of integral equations of Volterra and Volterra-Abel [11]. Thus, it is possible to find the analytical solution by the theory of the integral equations.

Let the components of a vector of speed ν and f_i, $(i = 1, \ldots, n)$ be the components of a given externally applied force satisfying conditions (1.3) and (3.1). Then, by transformation

$$\begin{cases} v_i = \lambda_i \vartheta(x_1, \ldots, x_n, t) + \Omega_i(x_1, \ldots, x_n, t)\exp\left(-\dfrac{t}{\delta_0\mu}\right), \quad (i = \overline{1, n}), \\ \Omega_i \equiv \dfrac{1}{2^n\sqrt{(\mu\pi t)^n}} \displaystyle\int_{R^n} \exp\left(-\dfrac{r^2}{4\mu t}\right) v_{0i}(s_1, \ldots, s_n)\, ds_1 \cdots ds_n, \quad (i = \overline{1, n}), \\ \vartheta|_{t=0} = 0; \quad \Omega_i|_{t=0} = v_{0i}(x_1, \ldots, x_n), \quad \forall (x_1, \ldots, x_n) \in R^n; \quad (i = \overline{1, n}), \end{cases}$$

(3.2)

the Navier-Stokes equation (1.1) becomes simpler because

$$\begin{cases} (x_1, \ldots, x_n) \in R^n, \quad t \in (0, T_0]: \\ v_{it} = \lambda_i \vartheta_t - \frac{1}{\delta_0 \mu} \Omega_i \exp\left(-\frac{t}{\delta_0 \mu}\right) + \Omega_{it} \exp\left(-\frac{t}{\delta_0 \mu}\right), \\ \Omega_{it} \equiv \frac{1}{\sqrt{\pi^n}} \int_{R^n} \frac{\sqrt{\mu}}{\sqrt{t}} \sum_{j=1}^{n} \tau_j \exp\left(-\left(\sum_{j=1}^{n} \tau_j^2\right)\right) v_{0il_j}(x_1 + 2\tau_1\sqrt{\mu t}, \ldots, x_n \\ \qquad + 2\tau_n \sqrt{\mu t}) d\tau_1 \cdots d\tau_n, \\ \qquad (l_j = x_j + 2\tau_j \sqrt{\mu t};\ s_j - x_j = 2\tau_j \sqrt{\mu t};\ j = \overline{1, n}), \\ v_{ix_j^m} = \lambda_i \vartheta_{x_j^m} + \exp\left(-\frac{t}{\delta_0 \mu}\right) \Omega_{ix_j^m}, \quad (m = 1, 2), \\ \Omega_{ix_j^2} \equiv \frac{1}{\sqrt{\pi^n}} \int_{R^n} \exp\left(-\left(\sum_{j=1}^{n} \tau_j^2\right)\right) v_{0il_j^2}(x_1 + 2\tau_1 \sqrt{\mu t}, \ldots, x_n + 2\tau_n \sqrt{\mu t}) d\tau_1 \cdots d\tau_n, \\ \mu \Delta \Omega_i \equiv \mu \frac{1}{\sqrt{\pi^n}} \int_{R^n} \exp\left(-\left(\sum_{j=1}^{n} \tau_j^2\right)\right) \sum_{j=1}^{n} v_{0il_j^2}(x_1 + 2\tau_i \sqrt{\mu t}, \ldots, x_n \\ \qquad + 2\tau_n \sqrt{\mu t}) d\tau_1 \cdots d\tau_n = |\text{integrating by parts}| = \frac{1}{\sqrt{\pi^n}} \int_{R^n} \frac{\sqrt{\mu}}{\sqrt{t}} \sum_{j=1}^{n} \tau_j \\ \qquad \times \exp(-(\tau_1^2 + \cdots + \tau_n^2)) v_{0il_j}(x_1 + 2\tau_1 \sqrt{\mu t}, \ldots, x_n + 2\tau_n \sqrt{\mu t}) d\tau_1 \cdots d\tau_n \equiv \Omega_{it}, \\ \qquad \text{i.e., } \Omega_{it} = \mu \Delta \Omega_i, \quad (\Omega_i |_{t=0} = v_{0i}(x_1, \ldots, x_n)), \\ \mu \Delta v_i = \mu \left\{ \lambda_i \Delta \vartheta + \exp\left(-\frac{t}{\delta_0 \mu}\right) \Delta \Omega_i \right\}, \\ v_{it} - \mu \Delta v_i = \lambda_i \vartheta_t - \frac{1}{\delta_0 \mu} \Omega_i \exp\left(-\frac{t}{\delta_0 \mu}\right) - \mu \lambda_i \Delta \vartheta, \quad (i = \overline{1, n}); \quad div\, \mathrm{v} = 0: \\ \sum_{i=1}^{n} \lambda_i \vartheta_{x_i} = 0; \quad \sum_{i=1}^{n} \Omega_{ix_i} = 0; \quad \sum_{j=1}^{n} \lambda_j \vartheta \lambda_i \vartheta_{x_j} = \lambda_i \vartheta \left(\sum_{j=1}^{n} \lambda_j \vartheta_{x_j} \right) = 0, \quad (i = \overline{1, n}), \\ \sum_{j=1}^{n} v_j v_{ix_j} \equiv \left(\sum_{j=1}^{n} \lambda_j \vartheta \Omega_{ix_j} + \sum_{j=1}^{n} \Omega_j \lambda_i \vartheta_{x_j} \right) \exp\left(-\frac{t}{\delta_0 \mu}\right) + \exp\left(-\frac{2t}{\delta_0 \mu}\right) \sum_{j=1}^{n} \Omega_j \Omega_{ix_j}. \end{cases} \tag{3.3}$$

Really, by (3.2) and (3.3), from the problem (1.1)-(1.3), we get

$$\lambda_i \vartheta_t + \left(\sum_{j=1}^{n} \lambda_j \vartheta \Omega_{ix_j} + \sum_{j=1}^{n} \Omega_j \lambda_i \vartheta_{x_j}\right) \exp\left(-\frac{t}{\delta_0 \mu}\right) + \exp\left(-\frac{2t}{\delta_0 \mu}\right) \sum_{j=1}^{n} \Omega_j \Omega_{ix_j}$$

$$= f_i - \frac{1}{\rho} P_{x_i} + \frac{1}{\delta_0 \mu} \Omega_i(x_1, \ldots, x_n, t) \exp\left(-\frac{t}{\delta_0 \mu}\right) + \mu \lambda_i \Delta \vartheta, \quad (i = \overline{1, n}). \quad (3.4)$$

From system (3.4), by APS and conditions (3.1), (3.3), we obtain

$$\begin{cases} \frac{1}{\rho} \Delta P = -\theta_n \left\{ F_0 + \exp\left(-\frac{t}{\delta_0 \mu}\right) B[\vartheta_{x_1}, \ldots, \vartheta_{x_n}] \right\}, \quad (n \geq 3), \\ \frac{1}{\rho} P = \int_{R^n} \frac{1}{r^{n-2}} \left\{ F_0(s_1, \ldots, s_n, t) \right. \\ \qquad \left. + (B[\vartheta_{s_1}, \ldots, \vartheta_{s_n}])(s_1, \ldots, s_n, t) \exp\left(-\frac{t}{\delta_0 \mu}\right) \right\} ds_1 \cdots ds_n, \\ \frac{1}{\rho} P_{x_i} = \int_{R^n} \frac{\tau_i (n-2)}{\sqrt{(\tau_1^2 + \cdots + \tau_n^2)^n}} \left\{ F_0(x_1 + \tau_1, \ldots, x_n + \tau_n; t) + \left(\exp\left(-\frac{t}{\delta_0 \mu}\right) \right) \right. \\ \qquad \left. \times (B[\vartheta_{h_1}, \ldots, \vartheta_{h_n}])(x_1 + \tau_1, \ldots, x_n + \tau_n; t) \right\} d\tau_1 \cdots d\tau_n, \\ B[\vartheta_{x_1}, \ldots, \vartheta_{x_n}] \equiv (\theta_n)^{-1} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_j \Omega_{ix_j} \right) \vartheta_{x_i} + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \Omega_{jx_i} \vartheta_{x_j} \right) \lambda_i \right), \\ F_0 \equiv (\theta_n)^{-1} \left[-\sum_{i=1}^{n} f_{ix_i} + \exp\left(-\frac{2t}{\delta_0 \mu}\right) \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \Omega_j \vartheta_{ix_j} \right) \right) \right], \\ \theta_n = 2(n-2)[\Gamma(2^{-1} n)]^{-1} \sqrt{\pi^n}; \quad (s_i - x_i = \tau_i;\ h_i = x_i + \tau_i;\ i = \overline{1, n}), \\ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(-\frac{1}{\rho} P_{x_i} \right) \equiv -\frac{1}{\rho} \Delta P; \quad \frac{\partial}{\partial t} \left[\sum_{i=1}^{n} \lambda_i \vartheta_{x_i} \right] = 0; \quad \mu \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\lambda_i \Delta \vartheta) = 0, \\ \left(\sum_{j=1}^{n} \Omega_{jx_j} \right)_{x_i} = 0, \quad \left(\sum_{j=1}^{n} \lambda_j \vartheta_{x_j} \right)_{x_i} = 0, \quad (i = \overline{1, n}), \\ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \sum_{j=1}^{n} \lambda_j \vartheta \Omega_{ix_j} + \sum_{j=1}^{n} \Omega_j \lambda_i \vartheta_{x_j} \right\} \\ = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_j \Omega_{ix_j} \right) \vartheta_{x_i} + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \Omega_{jx_i} \vartheta_{x_j} \right) \lambda_i. \end{cases}$$

(3.5)

This is a distribution law of pressure in the form of (3.5). At first, the similar results were obtained in the paper [7]. By (3.5), the system (3.4) is equivalently transformed to

$$\begin{cases} \vartheta_t = \Phi(x_1, \ldots, x_n, t) + \exp\left(-\dfrac{t}{\delta_0\mu}\right)\omega(x_1, \ldots, x_n, t) + \mu\Delta\vartheta, \\ \Phi \equiv d_0^{-1}\left(\displaystyle\sum_{i=1}^{n}\Omega_i\right)\dfrac{1}{\delta_0\mu}\exp\left(-\dfrac{t}{\delta_0\mu}\right) + \Phi_1(x_1, \ldots, x_n, t); \quad d_0 = \displaystyle\sum_{i=1}^{n}\lambda_i > 0, \\ \Phi_1 \equiv f_0 + d_0^{-1}\left[\displaystyle\sum_{i=1}^{n}\varphi_{i\delta_0} - \exp\left(-\dfrac{2t}{\delta_0\mu}\right)\displaystyle\sum_{i=1}^{n}\left(\displaystyle\sum_{j=1}^{n}\Omega_j\Omega_{ix_j}\right)\right. \\ \qquad \left. - \displaystyle\int_{R^n}\left\{\displaystyle\sum_{i=1}^{n}\dfrac{\tau_i(n-2)}{\sqrt{(\tau_1^2 + \cdots + \tau_n^2)^n}}(F_0(x_1 + \tau_1, \ldots, x_n + \tau_n; t))\right\}d\tau_1 \cdots d\tau_n\right], \\ \omega \equiv -\left\{d_0^{-1}\vartheta\left[\displaystyle\sum_{i=1}^{n}\left(\displaystyle\sum_{j=1}^{n}\lambda_j\Omega_{ix_j}\right)\right] + \displaystyle\sum_{j=1}^{n}\vartheta_{x_j}\Omega_j\right. \\ \qquad + d_0^{-1}\left[\displaystyle\int_{R^n}\left(\displaystyle\sum_{i=1}^{n}\dfrac{\bar{\tau}_i(n-2)}{\sqrt{(\tau_1^2 + \cdots + \tau_n^2)^n}}\right.\right. \\ \qquad \left.\left.\left.\times\{(B[\vartheta_{\bar{h}_1}, \ldots, \vartheta_{\bar{h}_n}])(x_1 + \bar{\tau}_1, \ldots, x_n + \bar{\tau}_n; t)\}\right)d\bar{\tau}_1 \cdots d\bar{\tau}_n\right]\right\}, \\ \qquad (\bar{h}_1 = x_i + \bar{\tau}_i, \; i = \overline{1, n}). \end{cases} \tag{3.6}$$

The problem (3.6) leads to the system of the integrated equations

$$
\begin{cases}
\vartheta = \Upsilon + \dfrac{1}{2^n\sqrt{\pi^n}}\displaystyle\int_0^t\int_{R^n}\left(\exp\left(-\frac{r^2}{4\mu(t-s)}\right)\right)\omega(s_1,\, \ldots,\, s_n,\, s)\exp\left(-\frac{s}{\delta_0\mu}\right) \\
\qquad \times\dfrac{ds_1\cdots ds_n ds}{(\sqrt{\mu(1-s)})^n} \\
\quad = \Upsilon + \dfrac{1}{\sqrt{\pi^n}}\displaystyle\int_0^t\int_{R^n}\exp\left(-\left(\sum_{i=1}^n \tau_i^2\right)\right)\omega(x_1 + 2\tau_1\sqrt{\mu(t-s)},\, \ldots,\, x_n \\
\qquad + 2\tau_n\sqrt{\mu(t-s)};\, s)\exp\left(-\dfrac{s}{\delta_0\mu}\right)d\tau_1\cdots d\tau_n ds \equiv (\Gamma\omega)(x_1,\, \ldots,\, x_n,\, t), \\
\omega = -\left\{d_0^{-1}(\Gamma\omega)\left[\displaystyle\sum_{i=1}^n\left(\sum_{j=1}^n \lambda_j\Omega_{ix_j}\right)\right]\right] + \displaystyle\sum_{j=1}^n(\Gamma_j\omega)\Omega_j \\
\qquad + d_0^{-1}\left[\displaystyle\int_{R^n}\left(\sum_{i=1}^n\frac{\bar\tau_i(n-2)}{(\bar\tau_1^2 + \cdots + \bar\tau_n^2)^n}\right.\right. \\
\qquad \left.\left.\times\,\{(B[\Gamma_1\omega,\, \ldots,\, \Gamma_n\omega])(x_1 + \bar\tau_1,\, \ldots,\, x_n + \bar\tau_n;\, t)\}\right)d\bar\tau_1\cdots d\bar\tau_n\right]\Bigg]\Bigg\} \\
\quad \equiv (\Gamma_0\omega)(x_1,\, \ldots,\, x_n,\, t), \\
\Upsilon(x_1,\, \ldots,\, x_n,\, t) \equiv \dfrac{1}{\sqrt{\pi^n}}\displaystyle\int_0^t\int_{R^n}\exp\left(-\left(\sum_{i=1}^n\tau_i^2\right)\right)\Phi(x_1 + 2\tau_1\sqrt{\mu(t-s)},\, \ldots,\, x_n \\
\qquad\qquad + 2\tau_n\sqrt{\mu(t-s)};\, s)d\tau_1\cdots d\tau_n ds, \\
\qquad\qquad (s_i - x_i = 2\pi_i\sqrt{\mu(t-s)};\, i = \overline{1,\, n}),
\end{cases}
\tag{3.7}
$$

where

$$\begin{cases} \vartheta_{x_i} = \Upsilon_{x_i} + \dfrac{1}{2^n\sqrt{\pi^n}}\displaystyle\int_0^t\int_{R^n}\left(\exp\left(-\frac{r^2}{4\mu(t-s)}\right)\right)\frac{-(x_i-s_i)}{2\mu(t-s)}\omega(s_1, \ldots, s_n, s) \\ \qquad \times\exp\left(-\dfrac{s}{\delta_0\mu}\right)\times\dfrac{ds_1\cdots ds_n ds}{(\sqrt{\mu(t-s)})^n} \\ \quad = \Upsilon_{x_i} + \dfrac{1}{\sqrt{\pi^n}}\displaystyle\int_0^t\int_{R^n}\exp\left(-\left(\sum_{i=1}^n\tau_i^2\right)\right)\omega(x_1+2\tau_1\sqrt{\mu(t-s)}, \ldots, x_n \\ \qquad + 2\tau_n\sqrt{\mu(t-s)};\, s)\exp\left(-\dfrac{s}{\delta_0\mu}\right)\dfrac{\tau_i d\tau_1\cdots d\tau_n ds}{\sqrt{\mu(t-s)}} \\ \quad \equiv (\Gamma_i\omega)(x_1, \ldots, x_n, t)(i=\overline{1, n}), \\ B[\Gamma_1\omega, \ldots, \Gamma_n\omega] \equiv (\theta_n)^{-1}\left(\displaystyle\sum_{i=1}^n\left(\sum_{j=1}^n\lambda_j\Omega_{ix_i}\right)\Gamma_i\omega + \sum_{i=1}^n\left(\sum_{j=1}^n\Omega_{jx_i}\Gamma_j\omega\right)\lambda_i\right), \\ \omega(x, t)\in C^{1,0}(T), \quad [x\in R^n, t\in[0, T_0];\, C^{1,0}(T)\equiv C^{\overbrace{1, \ldots, 1}^{n};\, 0}(T)]. \end{cases}$$

The system (3.7) consists of the integral equations on a variable $(x_1, \ldots, x_n, t)\in T$. We need to establish that the precompactness [11] of the family of (3.7) solutions satisfying, in other words, from any sequence, we can extract a subsequence that converges (in a suitable topology) to a solution to (3.7). This is the property required in order to deduce the strong convergence of approximate solutions to (3.7) and eventually, establish the existence of actual solutions. On the other hand, several standard methods are available for the construction of approximate solutions, one can for instance use Picard's method and we refer to paper [11] for further details concerning system (3.7).

Really, since δ_0 is small enough number, the operator Γ_0 satisfies conditions of a principle of compression. Then the solution of system ((3.7), the second equation) $\omega = \Gamma_0\omega$, can be found by Picard's method, that is,

$$\begin{cases} \Gamma_0 : k_{\Gamma_0}(\sqrt{\delta_0}) < 1; \quad \Gamma_0 S_{r_1} \subset S_{r_1}, \\ S_{r_1}(\omega_0) = \{\omega : |\,\omega - \omega_0\,| \le r_1, \quad (x_1, \ldots, x_n, t) \times T\}, \\ \omega_{m+1} = \Gamma_0 \omega_m, \quad (m = 0, 1, \ldots), \\ \|\,\omega_{m+1} - \omega\,\|_C \le (k_{\Gamma_0})^{m+1} r_1 \xrightarrow[m\to\infty]{k_{\Gamma_0}<1} 0, \end{cases} \tag{3.8}$$

here

$$\begin{cases} |\,\Gamma\omega\,| \le M_0 + \delta_0\mu\|\,\omega\,\|_C, \quad (\|\,\Upsilon_{x_i^m}\,\|_C \le M_i = \text{const}; \; m = 0, 1), \\ |\,\Gamma_i\omega\,| \le M_1 + \dfrac{1}{\sqrt{\pi^n}} \displaystyle\int_0^t \int_{R^n} \exp\left(-\left(\sum_{j=1}^n \tau_i^2\right)\right) |\,\tau_i\,| \exp\left(-\frac{s}{\delta_0\mu}\right) \frac{1}{\sqrt{\mu(t-s)}} \\ \qquad \times\, d\tau_1 \cdots d\tau_n ds \|\,\omega\,\|_C \le M_1 + C_1\sqrt{\delta_0}\,\|\,\omega\,\|_C, \quad (C_1 = 2^{-1}\pi e^{-\frac{1}{2}}\sqrt{n}), \\ \dfrac{1}{\sqrt{\pi^n}} \displaystyle\int_{R^n} \exp\left(-\left(\sum_{i=1}^n \tau_i^2\right)\right) |\,\tau_i\,| d\tau_1 \cdots d\tau_n \le \cdots \le \sqrt{2^{-1}n}, \\ \dfrac{1}{\sqrt{\mu}} \displaystyle\int_0^t \frac{1}{\sqrt{t-s}} \exp\left(-\frac{s}{\delta_0\mu}\right) ds = \int_0^t \frac{1}{\sqrt{t-s}} \{\sqrt{s(\delta_0\mu)^{-1}}\,[\exp(-\sqrt{(s(\delta_0\mu)^{-1})^2})]\} \\ \qquad \times \sqrt{\delta_0 s^{-1}} ds \le e^{-\frac{1}{2}}\pi\sqrt{2^{-1}\delta_0}, \\ (\sup\limits_{\psi\ge 0} \psi e^{-\psi^2} = e^{-\frac{1}{2}}\sqrt{2^{-1}}; \; \psi \equiv \sqrt{t(\delta_0\mu)^{-1}}), \\ \|\,\omega\,\|_C \le (1 - k_{\Gamma_0})^{-1} M_2, \quad (0 < M_2 = \text{const}). \end{cases}$$

Hence, by ((3.7), the first equation) $\vartheta = \Gamma\omega$, it follows:

$$\begin{cases} \vartheta_m = \Gamma\omega_m, \quad (m = 0, 1, \ldots; \; C_2 = \delta_0\mu < \delta_0; \; 0 < \mu < 1), \\ \|\,\vartheta_m - \vartheta\,\|_C \le C_2\|\,\omega_m - \omega\,\|_C \le C_2(k_{\Gamma_0})^m r_1 \xrightarrow[m\to\infty]{k_{\Gamma_0}<1} 0. \end{cases} \tag{3.9}$$

By taking into account the over established iteration process and the corresponding inequalities, one can state the following, i.e., the inequalities (3.8) and (3.9), assure the system (3.7) solution uniqueness.

Really, let us suppose that the system (3.7) does possess not only the solution $\{\vartheta, \omega\}$ but also another one $\{\overline{\vartheta}, \overline{\omega}\}$, where $\overline{\vartheta} = \Gamma\,\overline{\omega}$ and $\overline{\omega} = \Gamma_0\overline{\omega}$. It follows, by putting

$$\begin{cases} \| \omega - \overline{\omega} \|_C \le k_{\Gamma_0} \| \omega - \overline{\omega} \|, \quad (1 - k_{\Gamma_0} > 0), \\ \| \vartheta - \overline{\vartheta} \|_C \le C_2 \| \omega - \overline{\omega} \|_C, \end{cases}$$

which is satisfied if and only if $\| \omega - \overline{\omega} \|_C = 0,$ i.e., $\omega \equiv \overline{\omega};\ \vartheta \equiv \overline{\vartheta}.$

Further, on the basis of (3.2), the sequence $\{v_{i,m}\}_0^\infty$ converges to v_i,

$$\begin{cases} v_{i,m} = \lambda_i \vartheta_m + \Omega_i(x_1, ..., x_n, t) \exp\left(-\dfrac{t}{\delta_0 \mu}\right) \equiv A_i \vartheta_m, \quad (i = \overline{1, n};\ m = 0, 1, ...), \\ \| v_{i,m} - v_i \|_C \le \lambda_i \| \vartheta_m - \vartheta \|_C \le \lambda_i C_2 (k_{\Gamma_0})^m r_1 \xrightarrow[m \to \infty]{k_{\Gamma_0} < 1} 0, \quad (v_i \in \tilde{G}^1(D_0)). \end{cases}$$

(3.10)

But if the successions' $\{\vartheta_m\}_0^\infty$ and $\{v_{i,m}\}_0^\infty$ convergence conditions have been studied, then the assertion $\lim\limits_{m \to \infty} A_i \vartheta_m,\ (i = \overline{1, n})$ belongs to the classes of functions $\tilde{G}^1(D_0).$ Thus, we have

Theorem 2. *Consider the Navier-Stokes system* (1.1) *posed on the* $T = R^n \times [0, T_0]$ *and with prescribed initial data* (3.1) *and conditions* (3.8)-(3.10). *Then there exists the unique solution of the system* (3.7) *in* $\tilde{G}^1(D_0).$ *Therefore, by* (3.2), *the Navier-Stokes system has the unique solution in* $G_n^1(D_0).$

Remark 1. The similar results are take place for fluid with viscosity, when $\varphi_{i\delta_0} \equiv 0.$ In this case, $0 < \delta_0 < 1$ (see (3.2)), it is entered taking into account a condition (3.8).

4. Conclusions

The offered analytical methods of solution of nD Navier-Stokes problem transform this problem to inhomogeneous linear equations of heat conductivity type under Cauchy condition with enough smooth initial conditions, without attraction of any additional conditions. The analytical solutions of transformed equations are regular with respect to viscosity factor and simplify analysis of the initial problem in mathematical sense.

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